MULTIPLICATIVE MAPS ON IDEALS OF OPERATORS WHICH ARE LOCAL AUTOMORPHISMS

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ABSTRACT. We present the following reflexivity-like result concerning the automorphism group of the C^*-algebra $B(H)$, H being a separable Hilbert space. Let $\phi : B(H) \to B(H)$ be a multiplicative map (no linearity or continuity is assumed) which can be approximated at every point by automorphisms of $B(H)$ (these automorphisms may, of course, depend on the point) in the operator norm. Then ϕ is an automorphism of the algebra $B(H)$.

1. INTRODUCTION

Motivated by a problem of Larson [8, Some concluding remarks (5), p. 298] and a result of Kadison [7], in a series of papers we investigated the problem of the reflexivity of the automorphism groups of various operator algebras (see [2], [10], [11] and the references therein as well). This concerns the following question. Let $\mathcal{A}$ be a C^*-algebra (or, more generally, a Banach algebra). Let $\phi : \mathcal{A} \to \mathcal{A}$ be any bounded linear transformation. Suppose that for every $A \in \mathcal{A}$ there exists a sequence (ϕ_n) of automorphisms of $\mathcal{A}$ such that $\phi(A) = \lim_n \phi_n(A)$. Does it follow that ϕ is an automorphism? If the answer is 'yes', then we say that the automorphism group of $\mathcal{A}$ is topologically reflexive. If the condition that for every $A \in \mathcal{A}$ there exists an automorphism $\phi_A(A)$ of $\mathcal{A}$ such that $\phi(A) = \phi_A(A)$ implies that ϕ is an automorphism, then the automorphism group of $\mathcal{A}$ is called algebraically reflexive. In both cases, that is when speaking about topological or algebraic reflexivity, we have assumed that the map ϕ is linear and tried to conclude that then its multiplicativity and bijectivity follow. It is now a natural question that what happens if we assume the multiplicativity of ϕ and try to get its linearity and bijectivity. The interesting problem that when does the mutiplicativity imply additivity has been investigated in several papers. See, for example, [4], [15] which treat the additivity of semigroup automorphisms (we emphasize that bijectivity is assumed there) of operator algebras and [9]

for a general algebraic result on surjective semigroup homomorphisms onto prime rings. The aim of this note is to present the multiplicative counterpart of our recent topological reflexivity result for the automorphism group of $B(H)$ obtained in [2].

Let us now fix the notation and the concepts that we shall use throughout. If $\mathcal{A}$ is a *-algebra, then by an automorphism of $\mathcal{A}$ we mean a linear and multiplicative bijection of $\mathcal{A}$ onto itself which preserves the *-operation. In what follows let H be a separable Hilbert space and denote by $B(H)$ the C^*-algebra of all bounded linear operators acting on H. The ideal of finite rank operators and that of compact operators on H are denoted by $F(H)$ and $C(H)$, respectively. It is a well-known result due to Calkin that for every nontrivial ideal $\mathfrak{I}$ of $B(H)$ we have $F(H) \subset \mathfrak{I} \subset C(H)$. If $x, y \in H$, then $x \otimes y$ stands for the operator defined as $(x \otimes y)(z) = \langle z, y \rangle x$ $(z \in H)$. By a conjugate-linear isometry we mean a conjugate-linear operator $U : H \to H$ for which $\|Ux\| = \|x\|$ $(x \in H)$. Clearly, we have $\langle Ux, Uy \rangle = \langle y, x \rangle$ $(x, y \in H)$. Projection means any self-adjoint idempotent in $B(H)$.

2. The Results

Let $\mathcal{A} \subset B(H)$ be a standard operator algebra, that is, a *-subalgebra of $B(H)$ which contains every finite rank operator. It is a folk result that the automorphisms of $\mathcal{A}$ are exactly the transformations of the form

$$A \longmapsto UAU^*,$$

where $U \in B(H)$ is a unitary operator.

The main result of the paper reads as follows.

Theorem. *Let $\mathfrak{I} \subset B(H)$ be an ideal and $\phi : \mathfrak{I} \to \mathfrak{I}$ be a multiplicative map (linearity or continuity is not assumed). Suppose that for every $A \in \mathfrak{I}$ there exists a sequence (ϕ_n) of automorphisms of $\mathfrak{I}$ such that $\phi(A) = \lim_n \phi_n(A)$ in the operator norm.*

(1) *If $\mathfrak{I} = B(H)$, then ϕ is an automorphism of $\mathfrak{I}$.*
(2) *If $\mathfrak{I} \subsetneq B(H)$, then there is a (linear) isometry U such that $\phi(A) = UAU^*$ $(A \in \mathfrak{I})$. In particular, ϕ is a (linear) *-endomorphism of $\mathfrak{I}$.*

After this we immediately have the following corollary.

Corollary. *If $F(H) \subsetneq \mathfrak{I} \subset B(H)$ is an ideal and $\phi : \mathfrak{I} \to \mathfrak{I}$ is a multiplicative map with the property that for every $A \in \mathfrak{I}$ there exists an automorphism ϕ_A such that $\phi(A) = \phi_A(A)$, then ϕ is an automorphism of $\mathfrak{I}$.*

Remark 1. Let $U \in B(H)$ be an arbitrary isometry. It is easy to see that there is a sequence (U_n) of unitaries such that $U_n \to U$ in the strong operator topology. By Banach-Steinhaus theorem it follows that $U_n A U_n^* \to UAU^*$ in the operator norm for every compact operator A. This shows that the operator U appearing in the statement (2) of our theorem really only an isometry and not unitary in general.

Remark 2. Clearly, to any isometry U and any finite dimensional subspace $M \subset H$ there exists a unitary operator $V \in B(H)$ such that $U_{|M} = V_{|M}$. This implies easily that for any isometry U, the map $A \mapsto UAU^*$ is a local automorphism of $F(H)$, that is, to every $A \in F(H)$ there is an automorphism ϕ_A of $F(H)$ such that $UAU^* = \phi_A(A)$. Consequently, if H is infinite dimensional, the conclusion of the previous corollary does not hold true for $\mathfrak{I} = F(H)$.

Remark 3. Observe that since $B(H)$ is a prime ring, from [9] we could get the additivity of any multiplicative map ϕ on $B(H)$ if we had supposed that ϕ is surjective. However, as it is seen, this was not the case above. Clearly, without that surjectivity assumption we do not have the additivity in general. Consider, for example, the transformation

$$A \longmapsto \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix}.$$

This shows that our results are far from being trivial.

Proof of the Theorem. Let $\mathfrak{I}$ and ϕ be as in the statement of the theorem. By the form of the automorphisms of $\mathfrak{I}$ it is obvious that $\phi(P)$ is a rank-one projection for every rank-one projection $P \in \mathfrak{I}$. We show that ϕ is homogeneous on the set of all rank-one operators. First, let P be a rank-one projection and $\lambda \in \mathbb{C}$. Since $\phi(P)$ is also rank-one, we compute

$$(1) \qquad \phi(\lambda P) = \phi(P)\phi(\lambda P)\phi(P) = \mu\phi(P)$$

with some $\mu \in \mathbb{C}$. On the other hand, by the local property of ϕ, it follows that there are unitary operators $U_n \in B(H)$ such that

$$\phi(\lambda P) = \lim_n U_n(\lambda P)U_n^* = \lambda \lim_n U_n P U_n^*,$$

that is, $\phi(\lambda P)$ is equal to λ times a rank-one projection. Comparing this to (1) we obtain $\lambda = \mu$ and hence $\phi(\lambda P) = \lambda\phi(P)$. Now, if A is an arbitrary rank-one operator, then choosing a rank-one projection P for which $PA = A$ we have

$$\phi(\lambda A) = \phi(\lambda PA) = \phi(\lambda P)\phi(A) = \lambda\phi(P)\phi(A) = \lambda\phi(PA) = \lambda\phi(A).$$

Let $x \in H$ be any vector. Since, by the local property of ϕ, $\phi(x \otimes x)$ is a self-adjoint rank-one operator, it follows that there exists a vector $Tx \in H$ such that $\phi(x \otimes x) = Tx \otimes Tx$. For arbitrary $0 \neq x, y \in H$ we have

$$\phi(x \otimes y) = 1/(\|x\|^2\|y\|^2)\phi(x \otimes x \cdot x \otimes y \cdot y \otimes y) =$$

$$1/(\|x\|^2\|y\|^2)\phi(x \otimes x)\phi(x \otimes y)\phi(y \otimes y) = c \cdot Tx \otimes Ty,$$

where c is a complex number (depending on x, y). Since ϕ preserves the operator norm (this follows from the local property of ϕ), we obtain that

$$\|x\|\|y\| = \|\phi(x \otimes y)\| = |c|\|Tx \otimes Ty\| = |c|\|Tx\|\|Ty\| = |c|\|x\|\|y\|,$$

that is, $|c| = 1$. Suppose that $\langle x, y \rangle \neq 0$. We infer

$$c^2 \cdot Tx \otimes Ty \cdot Tx \otimes Ty = \phi(x \otimes y)\phi(x \otimes y) = \phi(x \otimes y \cdot x \otimes y) =$$

$$\langle x, y \rangle \phi(x \otimes y) = \langle x, y \rangle c \cdot Tx \otimes Ty$$

from which it follows that $c\langle Tx, Ty \rangle = \langle x, y \rangle$. Therefore, $|\langle Tx, Ty \rangle| = |\langle x, y \rangle|$. Similarly, in the case when $\langle x, y \rangle = 0$ we compute

$$0 = \phi(0) = \phi(x \otimes y \cdot x \otimes y) = c^2 \cdot Tx \otimes Ty \cdot Tx \otimes Ty$$

which gives us that $\langle Tx, Ty \rangle = 0$. Consequently, we have

$$|\langle Tx, Ty \rangle| = |\langle x, y \rangle| \qquad (x, y \in H).$$

We now apply Wigner's unitary-antiunitary theorem. This celebrated theorem says that any function on H which preserves the absolute value of the inner product is of the form $\varphi(x)Ux$ ($x \in H$), where φ is a so-called phase function (i.e., φ is a complex valued function H of modulus 1) and U is an either linear or conjugate-linear isometry (see, for example, [1], [13], [14] as well as [12] for a new, algebraic approach to the result). We claim that in our case U is linear. To see this, suppose on the contrary that U is conjugate-linear. Taking into account that $\phi(x \otimes x) = Ux \otimes Ux$ (recall that $|\varphi(x)| = 1$), in this case we find that

$$(2) \qquad \langle y, x \rangle \phi(x \otimes y) = \phi(x \otimes x \cdot y \otimes y) = Ux \otimes Ux \cdot Uy \otimes Uy =$$

$$\langle Uy, Ux \rangle Ux \otimes Uy = \langle x, y \rangle Ux \otimes Uy.$$

Now, let $x, y, v, w \in H$ be such that $\langle x, y \rangle, \langle x, w \rangle, \langle v, w \rangle \neq 0$. We compute

$$\phi(x \otimes y \cdot v \otimes w) = \langle v, y \rangle \phi(x \otimes w) = \langle v, y \rangle \frac{\langle x, w \rangle}{\langle w, x \rangle} Ux \otimes Uw.$$

On the other hand, we have

$$\phi(x \otimes y \cdot v \otimes w) = \phi(x \otimes y)\phi(v \otimes w) =$$

$$\frac{\langle x, y \rangle}{\langle y, x \rangle} Ux \otimes Uy \frac{\langle v, w \rangle}{\langle w, v \rangle} Uv \otimes Uw = \frac{\langle x, y \rangle}{\langle y, x \rangle} \frac{\langle v, w \rangle}{\langle w, v \rangle} \langle y, v \rangle Ux \otimes Uw.$$

This implies that

$$\langle x, y \rangle \langle y, v \rangle \langle w, x \rangle \langle v, w \rangle = \langle y, x \rangle \langle v, y \rangle \langle x, w \rangle \langle w, v \rangle.$$

Since this obviously does not hold true for every possible choice of the vectors $x, y, v, w \in H$, it follows that the operator U cannot be conjugate-linear. Therefore, we have a linear isometry U on H such that $\phi(x \otimes x) = Ux \otimes Ux$ ($x \in H$). Similarly as in (2) we get that $\phi(x \otimes y) = Ux \otimes Uy$ if $\langle x, y \rangle \neq 0$. In the opposite case choose a unit vector $v \in H$ such that $\langle x, v \rangle, \langle y, v \rangle \neq 0$. We have

$$\phi(x \otimes y) = \phi(x \otimes v \cdot v \otimes y) = Ux \otimes Uv \cdot Uv \otimes Uy = Ux \otimes Uy.$$

Now, let P be a projection of rank n. Choose pairwise orthogonal rank-one projections $P_1, \ldots, P_n$ such that $P = P_1 + \cdots + P_n$. By the multiplicativity

and the local property of ϕ it follows that $\phi(P_1), \dots, \phi(P_n)$ are pairwise orthogonal rank-one projections. Since

$$\phi(P_i) = \phi(P)\phi(P_i)\phi(P) \le \phi(P)I\phi(P) = \phi(P) \qquad (i = 1, \dots, n),$$

we infer that

(3) $$\phi(P_1) + \cdots + \phi(P_n) \le \phi(P).$$

By the equality of the ranks of the operators appearing on the two sides of (3), it follows that we have in fact equality in the above inequality, that is,

$$\phi(P_1) + \cdots + \phi(P_n) = \phi(P).$$

Now, let A be an arbitrary finite rank operator. Let P be a finite rank projection such that $A = PA$. If $P_1, \dots, P_n$ are as above and $P_1 = x_1 \otimes x_1, \dots, P_n = x_n \otimes x_n$, then we see that

(4) $$\phi(A) = \phi(PA) = \phi(P)\phi(A) = \sum_{i=1}^{n} \phi(P_i)\phi(A) = \sum_{i=1}^{n} \phi(P_i A) =$$

$$\sum_{i=1}^{n} \phi(x_i \otimes A^* x_i) = \sum_{i=1}^{n} U x_i \otimes U A^* x_i = \sum_{i=1}^{n} U(x_i \otimes x_i) A U^* =$$

$$U\left(\sum_{i=1}^{n} x_i \otimes x_i\right) A U^* = U(PA)U^* = UAU^*.$$

If H is finite dimensional, then U is necessarily unitary and hence we are done in that case. So, in what follows let us suppose that H is infinite dimensional. From (4) we see that on $F(H)$, the map ϕ can be represented as

$$\phi(A) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \qquad (A \in F(H)).$$

Clearly, on the whole ideal $\mathfrak{I}$, ϕ can be written as

$$\phi(A) = \begin{bmatrix} \psi_{11}(A) & \psi_{12}(A) \\ \psi_{21}(A) & \psi_{22}(A) \end{bmatrix} \qquad (A \in \mathfrak{I}).$$

Let $A \in \mathfrak{I}$ and consider an arbitrary finite rank projection P. From the equality $\phi(AP) = \phi(A)\phi(P)$ we obtain $\psi_{11}(A)P = AP$, $\psi_{21}(A)P = 0$. Since P was arbitrary, it follows that $\psi_{11}(A) = A$ and $\psi_{21}(A) = 0$. Similarly, from the equality $\phi(PA) = \phi(P)\phi(A)$ we infer that $\psi_{12}(A) = 0$. Consequently, our map ϕ is of the form

(5) $$\phi(A) = \begin{bmatrix} A & 0 \\ 0 & \psi_{22}(A) \end{bmatrix} \qquad (A \in \mathfrak{I}),$$

where ψ_{22} is obviously multiplicative and it vanishes on the finite rank operators.

Up till this point $\mathfrak{I}$ has been an arbitrary ideal in $B(H)$. Suppose now that $\mathfrak{I}$ is proper, that is, $\mathfrak{I} \subsetneq B(H)$. By the separability of H, the elements

of $\mathfrak{I}$ are all compact operators. Let A be an arbitrary compact operator. Denote $s_n(A)$ the nth s-number of A which is the nth term in the decreasing sequence of the eigenvalues of the positive compact operator $|A|$, where each eigenvalue is counted according to its multiplicity. For a fixed $n \in \mathbb{N}$, denote $\|A\|_n = s_1(A) + \cdots + s_n(A)$. It is well-known that $\|.\|_n$ is a norm (somtimes called Ky Fan norm) on $C(H)$ (see [3, p. 48]). Therefore, we have

$$|\|A\|_n - \|B\|_n| \le \|A - B\|_n \le n\|A - B\|$$

for any compact operators A, B. By the local property of ϕ it now follows that $\|\phi(A)\|_n = \|A\|_n$ for every $A \in \mathfrak{I}$. This gives us that the s-numbers of $\psi_{22}(A)$ are all zero and hence $\psi_{22}(A) = 0$. Consequently, we have

$$\phi(A) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \qquad (A \in \mathfrak{I})$$

which yields $\phi(A) = UAU^*$ for every $A \in \mathfrak{I}$.

It remains to consider the case when $\mathfrak{I} = B(H)$. The function $\psi_{22} : B(H) \to B(H)$ appearing in (5) is a multiplicative map which vanishes on the set of all finite rank operators. Suppose that $\psi_{22} \ne 0$. It is easily seen that $\psi_{22}(P) \ne 0$ for any infinite rank projection P. Indeed, this follows from the fact that for any infinite rank projection P there is a coisometry W such that $WPW^* = I$. Choosing uncountably many infinite rank projections with the property that the product of any two of them has finite rank (see, e.g., [10, Proof of Theorem 1]) and taking their values under ψ_{22} we would get uncountably many pairwise orthogonal nonzero projections in $B(H)$. Since this is a contradiction, we obtain $\psi_{22} = 0$. So, just as in the case when $\mathfrak{I} \subsetneq B(H)$, we have $\phi(A) = UAU^* \ (A \in B(H))$. But due to the local property of ϕ we have $\phi(I) = I$. Therefore, the isometry U is in fact unitary. This completes the proof of the theorem. $\qquad \square$

Proof of the Corollary. It follows from our theorem that there is an isometry U such that $\phi(A) = UAU^* \ (A \in \mathfrak{I})$. Since $F(H) \subsetneq \mathfrak{I}$, there exists an operator $A \in \mathfrak{I}$ with dense range. By the local property of ϕ, $\phi(A)$ must also have dense range. This gives us that U is surjective. $\qquad \square$

To conclude, we feel that it would be an interesting question to study our 'multiplicative' reflexivity problem for algebras of continuous functions which represent another important class of C^*-algebras.

Remark 4. The referee of the paper kindly informed us about the article [6] where the multiplicative selfmaps of the matrix algebra $M_n(\mathbb{C})$ are completely determined. He advised us to try to use that result to reach our conclusion as well as to look for possible generalizations. Here, we deal with only this second suggestion.

Let us suppose that our multiplicative map $\phi : \mathfrak{I} \to \mathfrak{I}$ is continuous in the operator norm topology (this was not supposed in Theorem). Assume that for every $A \in \mathfrak{I}$ there exists a sequence (ϕ_n) of unstarred automorphisms of $\mathfrak{I}$

such that $\phi(A) = \lim_n \phi_n(A)$ in the operator norm topology. Then similarly to our theorem we have the following assertions.

(1) If $\mathfrak{I} = B(H)$, then ϕ is an algebra-automorphism of $\mathfrak{I}$.
(2) If $\mathfrak{I} \subsetneq B(H)$, then there are bounded linear operators T, S on H with $ST = I$ such that $\phi(A) = TAS$ $(A \in \mathfrak{I})$. In particular, ϕ is an algebra-endomorphism of $\mathfrak{I}$.

We briefly sketch the proof. First we recall that every algebra-automorphism of $\mathfrak{I}$ is of the form $A \mapsto TAT^{-1}$, where T is an invertible bounded linear operator on H. Similarly to the first part of the proof of our theorem, we can see that ϕ is homogeneous on the set of rank-one operators. One can verify that ϕ preserves the rank of the finite rank idempotents and thus we obtain that for every $n \in \mathbb{N}$, ϕ can be considered as a selfmap of the matrix algebra $M_n(\mathbb{C})$. Applying [6, Theorem 1] we can infer that ϕ is linear on $F(H)$. Consequently, ϕ is an algebra-endomorphism of $F(H)$. Since ϕ maps any operator of rank one into an operator of rank at most one, and preserves the rank of the idempotents, following the argument in [5] till the proof of Theorem 1.2, we obtain that ϕ is of the form $\phi(A) = TAS$, with some $T, S \in B(H)$. We should emphasize that this is the place where we need the continuity of ϕ. In fact, it is not too hard to give a rank-preserving algebra-endomorphism of $F(H)$ which cannot be written in the nice form above. Consider, for example, a nonsurjective isometry V on H and an arbitrary unbounded linear operator L whose range is orthogonal to that of V. Then one can check that the map

$$\sum_{i=1}^{n} x_i \otimes y_i \longmapsto \sum_{i=1}^{n} (Vx_i) \otimes ((V+L)y_i)$$

has the desired properties. Turning back to our original problem, observe that since ϕ preserves the idempotents, we have $\langle Tx, S^*y \rangle = 1$ if $\langle x, y \rangle = 1$. Consequently, $\langle Tx, S^*y \rangle = \langle x, y \rangle$ $(x, y \in H)$, that is, $ST = I$. If $\mathfrak{I}$ is a proper ideal, then $F(H)$ is dense in $\mathfrak{I}$. Thus, by the continuity of ϕ we have the form $\phi(A) = TAS$ on the whole $\mathfrak{I}$. It remains to consider the case when $\mathfrak{I} = B(H)$. But this can be treated very similarly to the corresponding part of the proof of Theorem.

We note that we feel that the complete working-out of the proof above would not be shorter than what we have seen in the case of our theorem.

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